OPTIMAL CONSUMPTION AND PORTFOLIO
CHOICE IN AN INCOMPLETE MARKET DRIVEN BY
A JUMP DIFFUSION PROCESS

BY
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A Dissertation Submitted to the Graduate School
in Partial Fulfillment of the Requirements for the Degree

Erasmus Mundus Master: Models and Methods of Quantitative Economics (QEM)

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August, 2009
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Acknowledgement

First of all I would like to thank European Commission for providing the excellent Erasmus Mundus Master program of Models and Methods of Quantitative Economics (QEM) and the bountiful scholarship. Through this project, I have experienced different cultures in Europe deeply and successfully developed some necessary professional skills in economic research. Secondly I want to thank my advisor, Prof. Dr. Frank Riedel, for his helpful, inspiring and generous guidance. Moreover, I appreciate my family in Taiwan, especially my mother, for their constant love and support for me. Finally, I am grateful to all my friends, no matter where they are: Taiwan, Europe and U.S., for their warm and fragrant friendship.
Abstract

In Chapter 1, we introduce the fundamental Black-Scholes model and formula for the option pricing. This model is proposed by Black, Scholes and Merton in the early 1970s. Since there exist some empirical puzzles, e.g. volatility smile, which Black-Scholes model could not explain sufficiently, some models are developed to overcome these difficulties. One of the models, say double exponential jump diffusion model which is discussed in Chapter 2, gives the analytical solutions to perpetual American options and some path-dependent options. In Chapter 3, we would investigate when the market is not complete, and the risky asset is driven by both Brownian motion and jump diffusion, how can a small investor make the optimal consumption and portfolio choice in this case. Finally, a summary for the thesis would be given in Chapter 4.
Chapter 1

Introduction to Black-Scholes Model

In this chapter, we will give an introduction to the Black-Scholes Model, which is the contribution of Fischer Black, Myron Scholes and Robert Merton in the early 1970s. This model is the major breakthrough in the pricing of stock options (Hull, 2006). Here we would basically follow the notations from the book “Arbitrage Theory in Continuous Time Finance” by Tomas Björk (Björk, 2004).

1.1 The Model

Assume that there are two kinds of assets in the market: riskless bond with price $B_t$ and risky stock with price $S_t$. Both $B_t$ and $S_t$ operate continuously in time. The process $B_t$ is the price of riskless bond, with dynamics

$$\frac{dB_t}{dt} = r(t)B_t,$$  \hspace{1cm} (1.1)

where $r$ is the short rate of interest. The process $S_t$ is the price of risky stock, with dynamics

$$\frac{dS_t}{dt} = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t d\bar{W}_t,$$  \hspace{1cm} (1.2)

where $\bar{W}$ is a Wiener process. Both $\alpha$ and $\sigma$ are deterministic functions. $\alpha$ is the local mean rate of return of $S$ and $\sigma$ is the volatility of $S$.

The famous Black-Scholes Model consists two assets with the following dynamics

$$dB_t = rB_t dt,$$ \hspace{1cm} (1.3)

$$dS_t = \alpha S_t dt + \sigma S_t d\bar{W}_t,$$ \hspace{1cm} (1.4)
where $r$, $\alpha$ and $\sigma$ are deterministic constants. Suppose we have $x$ units of riskless bond and $y$ units of risky stock, then the value process is defined as $V_t^h = xB_t + yS_t$, with the portfolio $h$ defined as $h = (x, y)$.

A self-financed portfolio $h$ is called an arbitrage on a financial market if

1. $V_0^h = 0$,
2. $P(V_T^h \geq 0) = 1$,
3. $P(V_T^h > 0) > 0$.

Note: $T$ is the maturity date. We assume that the financial market is free of arbitrage, namely it exists no such arbitrage possibilities.

Later we will discuss the European call option, which is one of the most important derivatives. European call option is a derivative with the exercise price $K > 0$ and the time of maturity $T$ on the underlying asset $S$. The holder of the option has the right but not necessarily, to buy one share of underlying stock with price $K$ only at the precise maturity date $T$.

Conversely, European put option gives the holder the right to sell underlying asset $S$, with price $K$ at maturity time $T$. There are also American call option and American put option, which are quite similar to their European counterparts, except that the American ones can be exercised at any date before maturity time $T$. These types of contracts defined by the underlying asset $S$, are called derivative instruments or contingent claims. Here we would give a formal definition for the contingent claim.

Let us consider a price process $S$ in a financial market. A contingent claim with the maturity time $T$, is any stochastic variable $\chi \in \mathcal{F}_T^S$. This means that the value of $\chi$ at time $t$ can be completely determined by the observations of the trajectory $\{S(u); 0 \leq u \leq t\}$. A contingent claim $\chi$ is called a simple claim, if it has the following form $\chi = \Phi(S_T)$, where $\Phi$ is called pay off function or contract function.

European call option is a simple contingent claim, since its pay off function $\Phi(S_T)$ has the form:

$$\Phi(S_T) = \max [S_T - K, 0].$$

(1.5)

Similarly, European put option has the pay off function $\overline{\Phi}(S_T)$:

$$\overline{\Phi}(S_T) = \max[K - S_T, 0].$$

(1.6)
Our goal is to derive the fair price for such a contingent claim. We denote the fair price of European call option by \( f \). Intuitively, the price of such an option depends on the expected pay off at maturity time \( T \); thus it depends on the expected price of the asset \( S \) at maturity time \( T \), namely \( S_T \).

However, to form such an expectation for \( S_T \), one must rely on the current price of the underlying asset \( S_t \). Therefore, one may guess \( f \) could be written as the function of time \( t \) and \( S_t \) as \( f(t, S_t) \). Before proceeding to derive \( f(t, S_t) \), we have to introduce a proposition, to show the arbitrage-free requirement for a self-financing portfolio.

**proposition 1.1.1.** Assume it exists a self-financing portfolio \( h \), such that the value process of this portfolio has the following form:

\[
dV^h(t) = k(t)V^h(t)dt,
\]

where \( k \) is an adapted process. If there is no arbitrage possibility in this market, we must have \( k(t) = r(t) \) for all \( t \). Otherwise, there is an arbitrage opportunity.

**Proof.** Let us suggest \( k \) and \( r \) are constants temporarily. Then it gives two cases:

- \( k > r \): At time \( t = 0 \), one can borrow \( M_0 \) from the bank, with the interest rate \( r \), and then invest this \( M_0 \) in the portfolio \( h \). Since the growth rate of the portfolio \( k \) is greater than the interest rate \( r \), so at any time \( t > 0 \), one would have positive wealth. However, the initial net investment is zero. Therefore this is an arbitrage.

- \( k < r \): Do everything in the previous case conversely, then one would find this is also an arbitrage.

For \( k(t) \) and \( r(t) \), we can deal them with similar manner. Hence, from above we know that if there exists no arbitrage opportunity, one must have \( k(t) = r(t) \). \( \square \)

### 1.2 Black-Scholes Equation

Suppose we are in the market with assets \( B_t \) and \( S_t \) described by the following dynamics

\[
\begin{align*}
    dB_t &= rB_t dt, \\
    dS_t &= \alpha S_t dt + \sigma S_t d\bar{W}_t,
\end{align*}
\]

(1.8) (1.9)
Assume the European call (or put) option has a fair price $f(t, S_t)$. We could try to figure out $f(t, S_t)$ by making the combinations of $S_t$, $B_t$ and $f(t, S_t)$ arbitrage free. Set we have $\varphi_t$ units of calls and $\eta_t$ units of the underlying assets. Then the value process is

$$V_t = \varphi_t f(t, S_t) + \eta_t S_t. \quad (1.10)$$

Let $(\varphi_t, \eta_t)$ be self-financing, and the differentiation of $V_t$ is given by

$$dV_t = \varphi_t df(t, S_t) + \eta_t dS_t. \quad (1.11)$$

According to Itô formula, we may calculate $df(t, S_t)$ as

$$df(t, S_t) = \left( f_t + \alpha f_s S_t + \frac{1}{2} \sigma^2 f_{ss} S^2_t \right) dt + \sigma f_s S_t d\bar{W}_t. \quad (1.12)$$

Substitute $dS_t$ from (1.9) and $df(t, S_t)$ from (1.12) into (1.11), then $dV_t$ could be written as

$$dV_t = \left[ \varphi_t \left( f_t + \alpha f_s S_t + \frac{1}{2} \sigma^2 f_{ss} S^2_t \right) + \eta_t \alpha S_t \right] dt + \left[ \varphi_t (\sigma f_s S_t) + \eta_t \sigma S_t \right] d\bar{W}_t.$$

Proposition 1.1.1. tells us that for non-arbitrage portfolio process $V_t$, one would have $dV_t = rV_t dt$. This requirement implies that

$$(\varphi_t f_s + \eta_t) \sigma S_t = 0. \quad (1.13)$$

Without loss of generality, we may take $\varphi_t = -1$ and $\eta_t = f_s$. Consequently,

$$dV_t = \left[ - \left( f_t + \frac{1}{2} \sigma^2 f_{ss} S^2_t \right) \right] dt. \quad (1.14)$$

Besides, from proposition 1.1.1., we have

$$dV_t = rV_t dt = r \left( \varphi_t f(t, S_t) + \eta_t S_t \right) dt = r(-f + f_s S_t) dt. \quad (1.15)$$

Equalize (1.14) and (1.15), one would obtain the following equation

$$f_t + r f_s S_t + \frac{1}{2} \sigma^2 f_{ss} S^2_t = r f. \quad (1.16)$$
In addition, we have the boundary condition that \( f(T, S_T) = \Phi(S_T) \), which means that the option price should be equal to the pay off at maturity date \( T \). The above results could be summarized in the following theorem.

**Theorem 1.2.1 (Black-Scholes Equation).** If the market is described by (1.8) and (1.9), then the fair pricing function \( f(t, S_t) \) of the simple contingent claims, say European options, is defined in the domain \([0, T] \times \mathbb{R}_+\). Moreover, \( f(t, S_t) \) satisfies the following equations.

\[
\begin{align*}
    f_t(t, S_t) + rS_tf_s(t, S_t) + \frac{1}{2}\sigma^2S^2_tf_{ss}(t, S_t) & = rf(t, S_t), \\
    f(T, S_T) & = \Phi(S_T).
\end{align*}
\]

(1.17) (1.18)

### 1.3 Black-Scholes Formula

In this section we fix \( t \) as the initial time and \( T \) as the maturity date. Besides, \( u \) is the variable such that \( t \leq u \leq T \). Let us consider again the following financial market:

\[
\begin{align*}
    dB_u & = rB_udu, \\
    dS_u & = \alpha S_udu + \sigma S_ud\bar{W}_u,
\end{align*}
\]

(1.19) (1.20)

with a simple contingent \( \chi = \Phi(S_T) \). From the previous section, we know that \( \Phi(S_T) \) has an arbitrage free pricing function \( f(t, S_t) \). Set the probability measure which governs (1.19) and (1.20) is \( P \), namely objective probability measure. We can also define another probability measure \( Q \), under which the \( S \)-process has the form

\[
\begin{align*}
    dS_u & = rS_udu + \sigma S_udW_u, \\
    S_t & = s.
\end{align*}
\]

(1.21) (1.22)

Note that \( \bar{W}_u \) is Wiener process under probability measure \( P \), i.e. \( P \)-Wiener process; \( W_u \) is Wiener process under probability measure \( Q \), namely \( Q \)-Wiener process. From (1.20), we have

\[
S_T = S_te^{(\alpha - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - \bar{W}_t)}.
\]

(1.23)

Similarly from (1.21), one can obtain

\[
S_T = S_te^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}.
\]

(1.24)
Set $W_u = W_u - \theta u$, this implies that $dW_u = dW_u - \theta du$. Thus $dS_u$ can be written as

$$
\begin{align*}
dS_u &= (\alpha - \sigma \theta) S_u du + \sigma S_u dW_u, \\
&= r S_u du + \sigma S_u dW_u.
\end{align*}
$$

Then we have $\alpha - \sigma \theta = r$, which tells us that the risk premium $\theta = \frac{\alpha - r}{\sigma}$. Let $\Psi$ satisfies the following condition

$$
\begin{align*}
d\Psi_u &= -\theta \Psi_u d\bar{W}_u, \\
\Psi_t &= 1.
\end{align*}
$$

After some calculation, we may have

$$
\Psi(T) = e^{-\frac{\theta^2}{2}(T-t)} \Phi\left(\frac{\bar{W}_T - \bar{W}_t}{\sqrt{T-t}}\right). 
(1.25)
$$

We may claim that $\frac{S_u}{B_u} \Psi_u = \bar{S}_u \Psi_u$ is a martingale under $P$ where $B_u = B_t e^{(u-t)r}$ is the price of the riskless asset. This is because that $d(\bar{S}_u \Psi_u)$ has only $d\bar{W}_u$-term but $du$-term.

$$
\begin{align*}
d(\bar{S}_u \Psi_u) &= d\bar{S}_u \cdot \Psi_u + \bar{S}_u \cdot d\Psi_u + d\bar{S}_u \cdot d\Psi_u, \\
&= (\sigma - \theta) \bar{S}_u \Psi_u d\bar{W}_u.
\end{align*}
$$

Then $\bar{S}_T \Psi_T$ is also a martingale under $P$. Take European call as an example, we have the pay off $\Phi(S_T) = (S_T - K)^+$. The related non-arbitrage pricing function $f(t, S_t)$ could be written as

$$
\begin{align*}
f(t, S_t) &= e^{-r(T-t)} \int_{-\infty}^{+\infty} (S_T - K)^+ \Psi_T P(Z) dZ, 
(1.26)
\end{align*}
$$

where $Z \equiv \bar{W}_T - \bar{W}_t$ and $P(Z)$ is the density function of $Z$, with the form that $P(Z) = \frac{\Phi(0, \sqrt{T-t})}{\sqrt{2\pi(T-t)}}$. In other words, $Z$ follows the normal distribution $\phi(0, \sqrt{T-t})$.

Substitute $S_T$ from (1.23), $\Psi_T$ from (1.25) and $P(Z)$ into (1.26), one would have

$$
\begin{align*}
f(t, S_t) &= e^{-r(T-t)} \int_{-\infty}^{+\infty} \left[S_t e^{(\alpha - \sigma^2/2)(T-t) + \sigma Z} - K\right]^+ e^{-\frac{\theta^2}{2}(T-t)} \Phi\left(\frac{\bar{W}_T - \bar{W}_t}{\sqrt{T-t}}\right) e^{-\frac{\theta^2}{2}(T-t)} dZ.
\end{align*}
$$
Let $\Delta t$ denote $(T - t)$ and $s$ denote $S_t$, we may rewrite $f(t, S_t)$ as

$$f(t, S_t) = \frac{e^{-r\Delta t}}{\sqrt{2\pi \Delta T}} \int_{-\infty}^{+\infty} \left[ se^{(\alpha - \frac{\sigma^2}{2})\Delta T + \sigma Z} - K \right] e^{-\frac{\sigma^2}{2} \Delta T - \theta Z} e^{-\frac{\sigma^2}{2} \Delta T} dZ,$$

\[= \frac{e^{-r\Delta t}}{\sqrt{2\pi \Delta T}} \int_{Z^*}^{+\infty} \left[ se^{(\alpha - \frac{\sigma^2}{2})\Delta T + \sigma Z} - K \right] e^{-\frac{\sigma^2}{2} \Delta T} (Z + \Delta T)^2 dZ,\]

where $Z^* = \frac{1}{\sigma} \left( \ln \frac{K}{s} - (\alpha - \frac{\sigma^2}{2}) \right)$. Then we change the integral variable from $Z$ to $Y$.

Let $Y$ be defined as $Y \triangleq \frac{Z + \theta \Delta T}{\sqrt{\Delta T}}$, and the lower integral limit $Z^*$ can be written as $Y^*$ with the form

$$Y^* = \frac{1}{\sigma \sqrt{\Delta T}} \left( \ln \frac{K}{s} - (r - \frac{\sigma^2}{2}) \Delta T \right).$$

With the above notations, we can continue the calculation of $f(t, S_t)$ as

$$f(t, S_t) = \frac{e^{-r\Delta t}}{\sqrt{2\pi \Delta T}} \int_{Y^*}^{+\infty} \left[ se^{(\alpha - \frac{\sigma^2}{2})\Delta T + \sigma \sqrt{\Delta T} Y} - K \right] e^{-\frac{\sigma^2}{2} \Delta T} \sqrt{\Delta T} dY,$$

\[= e^{-r\Delta t} \left[ \frac{1}{\sqrt{2\pi}} \int_{Y^*}^{+\infty} se^{(\alpha - \frac{\sigma^2}{2})\Delta T + \sigma \sqrt{\Delta T} Y} \cdot e^{-\frac{\sigma^2}{2} Y} dY \right] - \frac{1}{\sqrt{2\pi}} \int_{Y^*}^{+\infty} Ke^{-\frac{\sigma^2}{2} Y} dY.\]

Then we calculate (a)-term and (b)-term seperately.

\[\begin{align*}
(a) &= se^{(\alpha - \frac{\sigma^2}{2})\Delta T} \frac{1}{\sqrt{2\pi}} \int_{Y^*}^{+\infty} e^{-\frac{\sigma^2}{2} Y} dY, \\
&= se^{\Delta T} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-Y^* + \sigma \sqrt{\Delta T}} e^{-\frac{1}{2} (Y - \sigma \sqrt{\Delta T})^2} d(Y - \sigma \sqrt{\Delta T}), \\
&= se^{\Delta T} N(-Y^* + \sigma \sqrt{\Delta T}), \\
&= se^{\Delta T} N(d_1),
\end{align*}\]

where $d_1 \triangleq -Y^* + \sigma \sqrt{\Delta T} = \frac{1}{\sigma \sqrt{\Delta T}} \left( \ln \frac{K}{s} + (r + \frac{\sigma^2}{2}) \Delta T \right)$, and $N(\cdot)$ is the cumulative density function of standard normal distribution. We can deal (b)-term in a similar way.

\[\begin{align*}
(b) &= \frac{1}{\sqrt{2\pi}} \int_{Y^*}^{+\infty} Ke^{-\frac{\sigma^2}{2} Y} dY = K \frac{1}{\sqrt{2\pi}} \int_{Y^*}^{+\infty} e^{-\frac{\sigma^2}{2} Y} dY = K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-Y^*} e^{-\frac{\sigma^2}{2} Y} dY, \\
&= KN(-Y^*) = KN(d_1),
\end{align*}\]
in which $-Y^* = \frac{1}{\sigma \sqrt{\Delta T}} \left( \ln \frac{s}{K} + (r - \frac{\sigma^2}{2}) \Delta T \right) \triangleq d_2$. Substitute $(a)$-term and $(b)$-term into $f(t, S_t)$ and finally we may obtain $f(t, S_t)$ with the formula

$$f(t, S_t) = e^{-r\Delta T} \left( se^{r\Delta T} N(d_1) - KN(d_2) \right) = sN(d_1) - e^{-r\Delta T} KN(d_2).$$

(1.27)

Besides, European put option can be treated in a similar manner. We can summarize this section by the following proposition.

**proposition 1.3.1.** The non-arbitrage price of European call option with strike price $K$ and maturity time $T$, is the function $f(t, S_t)$ such that

$$f(t, S_t) = sN(d_1) - e^{-r(T-t)} KN(d_2),$$

(1.28)

with

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \frac{K}{s} + (r + \frac{\sigma^2}{2})(T-t) \right),$$

$$d_2 = \frac{1}{\sigma \sqrt{T-t}} \left( \ln \frac{K}{s} + (r - \frac{\sigma^2}{2})(T-t) \right).$$

Similarly, European put option has the non-arbitrage price $\tilde{f}(t, S_t)$ of the form

$$\tilde{f}(t, S_t) = e^{-r(T-t)} KN(-d_2) - sN(-d_1),$$

(1.29)

with $d_1$ and $d_2$ given above. This is the famous Black-Scholes formula for European call and put options.
Chapter 2

Introduction to the Double Exponential Jump Diffusion Model

As discussed in Chapter 1, Black-Scholes model is indeed a major breakthrough in the pricing of stock options. This model assumes that the volatility $\sigma$ should be a constant. However, it is a generally acknowledged fact that instead of being a constant, the curve of implied volatility versus strike price seems like a smile. Namely, the implied volatility is a convex curve versus the strike price. Moreover according to Black-Scholes model, the return should follow the log-normal distribution, nevertheless, the return distribution is skewed to the left plus a higher peak and two heavier tails than those of the normal distribution. These two empirical phenomena, namely the volatility smile and the asymmetric leptokurtic return distribution, could not be explained by the Black-Scholes model sufficiently.

Many authors have tried to tackle these problems by modifying the Black-Scholes model, for example, fractal Brownian motion by Black and Cox (1976) or time-changed Brownian motion. Some of them can provide analytical solutions to the European options. Yet when it comes to path-dependent options as perpetual American options and lookback options, most of them cannot have analytical pricing formula under these proposed models.

Nevertheless, Kou and Wang (2001, 2003, 2004) provide the double exponential jump diffusion model, under which even the path-dependent options can have exact solutions. In principle we would follow the methods and notations from Kou and Wang (2001) in this chapter, and show that how to price the perpetual American options in the double exponential jump diffusion model.
\subsection{The Model}

In the double exponential jump diffusion model, the dynamics of the stock price $S(t)$ is written as

$$\frac{dS(t)}{S(t^-)} = \mu dt + \sigma dW(t) + d \left( \sum_{i=1}^{N(t)} (V_i - 1) \right),$$  \hspace{1cm} (2.1)

where $S(t^-) = \lim_{u \to t} S(u)$ is the left-side limit. The first two terms in (2.1) are the familiar geometric Brownian motion, which represents the flow of diffusion information, with the mean rate of return $\mu$, the volatility constant $\sigma$ and the Wiener Process $W(t)$. While the last term in (2.1) denotes a jump part, with the number of jumps until time $t$ $N(t)$ and the jump size $V_i$. 

Note that $N(t)$ corresponds to Poisson process with the intensity $\lambda$. Besides, the logarithm of jump sizes $Y_i = \log(V_i)$ is an i.i.d. random variable whose density function has the asymmetric double exponential distribution. This is where the name of the model comes from. More precisely, the density function of $Y$ has the form

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{y \geq 0} + q \cdot \eta_2 e^{-\eta_2 y} 1_{y < 0}, \quad \eta_1 > 1, \eta_2 > 0,$$  \hspace{1cm} (2.2)

where $p \geq 0$, $q \geq 0$, and $p + q = 1$. While we calculate the expectation of stock price $S(t)$, we would encounter the term $\int e^{-\eta_1 y} \cdot e^{y} dy$ and to guarantee the finiteness of this term, $\eta_1$ is required to be strictly greater than 1. Assume that all the random variables here $W(t), Y(t)$ and $N(t)$ are independent from each other. Let us define the return process as $X(t) \triangleq \log \left( \frac{S(t)}{S(0)} \right)$, which is given by

$$X(t) = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0.$$  \hspace{1cm} (2.3)

Because the market is incomplete, there exists more than one risk-neutral probability measure. Yet according to Kou (2000), it gives a particular risk-neutral probability measure $P^*$, under which we may obtain the rational equilibrium price of an option by calculating the expectation of the discounted option payoff. Thus we may rewrite (2.1) under the risk-neutral probability measure $P^*$ as

$$\frac{dS(t)}{S(t^-)} = (r - \lambda^* \zeta^*) dt + \sigma dW^*(t) + d \left( \sum_{i=1}^{N(t)^*} (V_i^* - 1) \right).$$  \hspace{1cm} (2.4)
Similarly, the return process $X(t)$ can be modified by

$$X(t) = (r - \frac{\sigma^2}{2} - \lambda^* \zeta^*) t + \sigma W^*(t) + \sum_{i=1}^{N^*(t)} Y_i^*, \ X(0) = 0. \quad (2.5)$$

Note that all the random variables are defined under $P^*$. Moreover, the logarithm of jump size $\{Y_i^*\}$ for $i = 1, \ldots, N^*(t)$ would have the new density function

$$f_Y(y) = p^* \eta_1^* e^{-\eta_1^* y} 1_{y \geq 0} + q^* \eta_2^* e^{-\eta_2^* y} 1_{y < 0}, \ \eta_1^* > 1, \ \eta_2^* > 0, \quad (2.6)$$

with the constants $p^* \geq 0, q^* \geq 0, p^* + q^* = 1, \lambda^* > 0$, and

$$\zeta^* = E^*[V^*] - 1 = \frac{p^* \eta_1^*}{\eta_1^* - 1} + \frac{q^* \eta_2^*}{\eta_2^* + 1} - 1. \quad (2.7)$$

To make the calculations easier, we would drop $*$ in the later context as Kou and Wang (2001).

In Merton (1976), one can find that the logarithm of jump size $Y$ follows the normal distribution, and the model becomes the normal jump diffusion model. Noteworthily, both the double exponential jump diffusion model and the normal jump diffusion model are the special cases of Lévy process models, and a more general setting is found in Duffie, Pan, and Singleton (2000). In addition, the intuition why the double exponential jump diffusion models can give the analytical tractibility to the problems of option pricing, is that the products of exponential distributions remains to be an exponential distribution, and in many cases the integration of exponential distribution can be solved analytically. In the next section we would come up with the application of this model for perpetual American put option.

### 2.2 Pricing Formula of Perpetual American Options

In this section, we would derive the closed form of perpetual American options with infinite maturity. Although these infinite maturity options are not traded on real markets, they can still serve as examples to realize finance theory. Moreover, we can approximate the value of finite horizon American options with the solution of the perpetual American options. As in Kou and Wang (2001), we would only deal with the put option here, and the call option can be treated in a similar attitude.
From the previous section, the stock price $S(t)$ is given by

$$\frac{dS(t)}{S(t)} = (r - \lambda \zeta)dt + \sigma dW(t) + d \left( \sum_{i=1}^{N(t)} (V_i - 1) \right),$$

and the return process $X(t) = \log \left( \frac{S(t)}{S(0)} \right)$ behaves as

$$X(t) = (r - \frac{\sigma^2}{2} - \lambda \zeta)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i, \quad X(0) = 0.$$

Then the rational expectation equilibrium price of the perpetual American put is of the form

$$\psi(S(0)) = \sup_\tau E^* \left[ e^{-r\tau} (K - S(\tau))^+ \right] = \sup_\tau \left[ e^{-r\tau} (K - S(0)e^{X(\tau)})^+ \right],$$

which is the expectation of the discounted payoff of the perpetual American put under the risk-neutral probability measure $P^*$, and the supremum is taken over all stopping times $\tau$ in $[0, +\infty)$, while $K$ is the strike price. And the moment generating function of $X(t)$ is $G(\theta)$ of the form

$$E^* \left[ e^{\theta X(t)} \right] = \exp(G(\theta)t),$$

More precisely, $G(\theta)$ is defined as

$$G(x) \triangleq x(r - \frac{1}{2} \sigma^2 - \lambda \zeta) + \frac{1}{2} \sigma^2 + \lambda \left( \frac{p \eta_1}{\eta_1 - x} + \frac{q \eta_2}{\eta_2 + x} - 1 \right),$$

with the coefficient

$$\zeta = E^*[V] - 1 = \frac{p \eta_1}{\eta_1 - 1} + \frac{q \eta_2}{\eta_2 + 1} - 1.$$

From Lemma 2.1 in Kou and Wang (2003), we know that the equation

$$G(x) = \alpha \text{ for all } \alpha > 0$$

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has exactly four roots $\beta_1, \beta_2, -\beta_3, -\beta_4$ such that

$$0 < \beta_1 < \eta_1 < \beta_2 < +\infty, \quad 0 < \beta_3 < \eta_1 < \beta_4 < +\infty.$$ \hspace{1cm} (2.15)

These four roots are important since they would help us to characterize the solution of the perpetual American options which would be given in Theorem 2.2.2. Yet before we introduce this theorem, we would first come to Lemma 2.2.1.

Let the value of American put be $\psi(S(0)) = V(S(0))$ where $V$ is the value function. If $V$ satisfies the properties of Lemma 2.2.1, then Theorem 2.2.2. can be proved. Namely we may have analytical pricing formula for perpetual American put option. This is the reason why Lemma 2.2.1 is important.

**Lemma 2.2.1.** Assume that there exists some real number $x_0 < \log K$ and a non-negative $C^1$ function $u(x)$. Let the following properties hold.

1. $u(x)$ is convex and $C^2$ on $\mathbb{R} \setminus \{x_0\}$, with $u''(x_0^-)$ and $u''(x_0^+)$ existing.
2. $(\mathcal{L}u)(x) - ru(x) = 0, \forall x > x_0.$
3. $(\mathcal{L}u)(x) - ru(x) < 0, \forall x < x_0.$
4. $u(x) > (K - e^x)^+, \forall x > x_0.$
5. $u(x) = (K - e^x)^+, \forall x \leq x_0.$
6. there exists some random variable $Z$ with $E^*[Z] < \infty$, and

$$e^{-r(t \wedge \tau \wedge \tau^*)} u(X(t \wedge \tau \wedge \tau^*) + x) \leq Z, \forall t \geq 0, \forall x,$$ and every stopping time $\tau$.

Let the infinitesimal generator be defined as

$$\mathcal{L}V(x) \triangleq \frac{1}{2} \sigma^2 V''(x) + \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) V'(x) + \lambda \int_{-\infty}^{+\infty} [V(x + y) - V(x)] dF(y).$$ \hspace{1cm} (2.16)

Then under the above assumptions, the option price has the form

$$\psi(S(0)) = u(\log S(0)),$$ \hspace{1cm} (2.17)

and the optimal stopping time is given by

$$\tau^* \triangleq \inf \{ t \geq 0 ; S(t) \leq e^{x_0} \}.$$ \hspace{1cm} (2.18)
Proof. According to Kou and Wang (2001), they define \( \tilde{X}(t) = x + X(t) \). \( \tilde{X}(t) \) has the same infinitesimal generator \( \mathcal{L} \) with \( X(t) \), namely (2.16). Then Mordecki (1999, p. 230-232) provides a similar argument for this proof, however, \( M(t) \) has to be replaced by

\[
M(t) = e^{-rt}u(\tilde{X}(t)) - \int_0^t \left[ -ru(\tilde{X}(s)) + \mathcal{L}u(\tilde{X}(s)) \right] ds. \tag{2.19}
\]

With the help of Lemma 2.2.1, we may come up with Theorem 2.2.2.

**Theorem 2.2.2.** The value of American put option has the form \( \psi(S(0)) = V(S(0)) \), where the value function \( V \) is described as

\[
V(v) = \begin{cases} 
K - v & \text{if } v < v_0, \\
Ae^{-\beta_3 r} + Be^{-\beta_4 r} & \text{if } v \geq v_0,
\end{cases} \tag{2.20}
\]

with the following optimal exercise boundary, coefficients and optimal stopping time \( \tau^* \) respectively,

\[
v_0 = K \frac{\eta_2 + 1}{\eta_2} \cdot \frac{\beta_{3,r}}{1 + \beta_{3,r}} \cdot \frac{\beta_{4,r}}{1 + \beta_{4,r}}, \tag{2.21}
\]

\[
A = v_0 \frac{1 + \beta_{4,r}}{\beta_{4,r} - \beta_{3,r}} \left[ \frac{\beta_{4,r}}{1 + \beta_{4,r}} K - v_0 \right] > 0, \tag{2.22}
\]

\[
B = \frac{1 + \beta_{3,r}}{\beta_{4,r} - \beta_{3,r}} \left[ v_0 - \frac{\beta_{3,r}}{1 + \beta_{3,r}} K \right] > 0, \tag{2.23}
\]

\[
\tau^* = \inf \{ t \geq 0 : S(t) \leq v_0 \}. \tag{2.24}
\]

**Proof.** Let \( x_0 = \ln v_0 \) and the variable \( x = \ln v \), then the value function \( V \) can be rewritten as

\[
V(x) = \begin{cases} 
K - e^x & \text{if } x < x_0, \\
Ae^{-\beta_3 r x} + Be^{-\beta_4 r x} & \text{if } x \geq x_0.
\end{cases} \tag{2.25}
\]

Later \( \beta_{3,r} \) and \( \beta_{4,r} \) would be abbreviated as \( \beta_3 \) and \( \beta_4 \) respectively. Our goal is to show that \( V(x) \) satisfies Lemma 2.2.1., then we would have

\[
\Psi(S(0)) = u(\log S(0)) = \begin{cases} 
K - e^{\log S(0)} & \text{if } \log S(0) < \log v_0, \\
Ae^{-\beta_3 \log S(0)} + Be^{-\beta_4 \log S(0)} & \text{if } \log S(0) \geq \log v_0.
\end{cases} \tag{2.26}
\]

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Or equivalently,

\[ V(v) = \begin{cases} 
  K - v & \text{if } v < v_0, \\
  A v^{-\beta_3} + B v^{-\beta_4} & \text{if } v \geq v_0,
\end{cases} \tag{2.27} \]

with \( S(0) \) being replaced by \( v \). Thus the proof would be completed in this way.

First we would compute the term \((\mathcal{L}V)(x) - rV(x)\). In the case of \( x > x_0 \), we have

\[
(\mathcal{L}V)(x) - rV(x) = \frac{1}{2} \sigma^2 V''(x) + \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) V'(x) + \lambda \int_{-\infty}^{+\infty} [V(x + y) - V(x)] dF(y) - rV(x),
\]

\[
= \frac{1}{2} \sigma^2 V''(x) + \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) V'(x) + \lambda \int_{-\infty}^{+\infty} V(x + y) dF(y) - V(x) - rV(x).
\]

Then we may compute the terms \((a), (b)\) and \((c)'\) sequentially.

\[(a) = \frac{1}{2} \sigma^2 \left( A \beta_3^2 e^{-\beta_3 x} + B \beta_4^2 e^{-\beta_4 x} \right). \tag{2.28} \]

\[(b) = \left( r - \frac{1}{2} \sigma^2 - \lambda \zeta \right) \left( -A \beta_3 e^{-\beta_3 x} - B \beta_4 e^{-\beta_4 x} \right). \tag{2.29} \]

\[(c)'
= \int_{-\infty}^{x_0-x} V(x + y) dF(y) + \int_{x_0-x}^{0} V(x + y) dF(y) + \int_{0}^{\infty} V(x + y) dF(y) \tag{2.30} \]

\[
= \int_{-\infty}^{x_0-x} (K - e^{x+y}) q_{\beta_3} e^{\eta_2 y} dy + \int_{x_0-x}^{0} (A e^{-\beta_3(x+y)} - B e^{-\beta_4(x+y)}) q_{\eta_1} e^{\eta_1 y} dy \\
+ \int_{0}^{\infty} (A e^{-\beta_3(x+y)} - B e^{-\beta_4(x+y)}) p_{\eta_1} e^{-\eta_1 y} dy. \tag{2.31} \]
After some simplification, we have

\[
(c)' = q\eta_2 e^{\eta_2(x_0-x)} \left( K - \frac{\eta_2 e^{x_0}}{1+\eta_2} \right) + \frac{q\eta_2 A}{\eta_2 - \eta_3} \left( e^{-\beta_3 x} - e^{-\beta_3 x_0} \cdot e^{(x_0-x)\eta_2} \right) + \frac{q\eta_2 B}{\eta_2 - \eta_4} \left( e^{-\beta_4 x} - e^{-\beta_4 x_0} \cdot e^{(x_0-x)\eta_2} \right) + \left[ A\eta_1 e^{-\beta_3 x} \eta_1 + \beta_3 \right] + B\eta_1 e^{-\beta_4 x} \eta_1 + \beta_4 \right] \tag{2.32}
\]

Then \( (c) = (c)' - V(x) = (c)' - (Ae^{-\beta_3 x} + Be^{-\beta_4 x}) \). With the help of \( (a), (b) \) and \( (c) \), we may compute \( \mathcal{L}V(x) - rV(x) \) as follows.

\[
\mathcal{L}V(x) - rV(x) = (a) + (b) + \lambda \cdot (c) - rV(x),
\]

\[
= Ae^{-\beta_3 x} \left( \frac{1}{2}\sigma^2 - \beta_3 \left( r - \frac{\sigma^2}{2} - \lambda\zeta \right) + \frac{\lambda q\eta_2}{\eta_2 - \beta_3} + \frac{\lambda p\eta_1}{\eta_1 + \beta_3} - \lambda - r \right) + Be^{-\beta_4 x} \left( \frac{1}{2}\sigma^2 - \beta_4 \left( r - \frac{\sigma^2}{2} - \lambda\zeta \right) + \frac{\lambda q\eta_2}{\eta_2 - \beta_4} + \frac{\lambda p\eta_1}{\eta_1 + \beta_4} - \lambda - r \right) + \lambda qe^{(x_0-x)\eta_2} \left( K - \frac{\eta_2 e^{x_0}}{1+\eta_2} - \frac{\eta_2 A e^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 B e^{-\beta_4 x_0}}{\eta_2 - \beta_4} \right), \tag{2.33}
\]

\[
= Ae^{-\beta_3 x} f(\beta_3) + Be^{-\beta_4 x} f(\beta_4),
\]

\[
+ \lambda qe^{(x_0-x)\eta_2} \left( K - \frac{\eta_2 e^{x_0}}{1+\eta_2} - \frac{\eta_2 A e^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 B e^{-\beta_4 x_0}}{\eta_2 - \beta_4} \right). \tag{2.34}
\]

where

\[
f(x) \triangleq G(-x) - r = \frac{1}{2} e^2 \sigma^2 - x \left( r - \frac{1}{2} \sigma^2 - \lambda\zeta \right) - r + \lambda \left( \frac{q\eta_2}{\eta_2 - x} + \frac{p\eta_1}{\eta_1 + x} - 1 \right). \tag{2.35}
\]

Similarly for the case of \( x < x_0 \), we may have

\[
\mathcal{L}V(x) - rV(x) = -rK - \lambda qe^{-(x-x_0)\eta_1} \left( K - \frac{\eta_1 e^{x_0}}{\eta_1 - 1} - \frac{\eta A e^{-\beta_3 x_0}}{\eta_1 + \beta_3} - \frac{\eta B e^{-\beta_4 x_0}}{\eta_1 + \beta_4} \right). \tag{2.36}
\]

Now we can continue to check if the value function \( V(x) \) satisfies the six conditions in Lemma 2.2.1.

1. Check if \( V(x) \) is \( C^2 \) on \( \mathbb{R} \setminus \{x_0\} \), and is convex with \( V'(x_{0-}) \) and \( V''(x_{0+}) \)
existing. We may calculate $V''(x_0^-)$ and $V''(x_0^+)$ as follows.

$$V''(x_0^-) = -e^{x_0}, \quad (2.37)$$

$$V''(x_0^+) = A\beta_3^2 e^{-\beta_3 x_0} + B\beta_4^2 e^{-\beta_4 x_0}. \quad (2.38)$$

From above we know that both $V''(x_0^-)$ and $V''(x_0^+)$ exist. Besides $V(x)$ is $C^2$ on $\mathbb{R} \setminus \{x_0\}$ with $V''(x_0^-) \neq V''(x_0^+)$. Moreover, $V(x)$ is a convex function. Therefore condition 1. is satisfied.

2. Check if $(\mathcal{L}V)(x) - rV(x) = 0, \forall x > x_0$. For the case $x > x_0$, we have

$$(\mathcal{L}V)(x) - rV(x) = Ae^{-\beta_3 x} f(\beta_3) + Be^{-\beta_4 x} f(\beta_4),$$

$$+ \lambda q e^{(x_0-x)\eta_2} \left(K - \frac{\eta_2 e^{x_0}}{1 + \eta_2} - \frac{\eta_2 A e^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 B e^{-\beta_4 x_0}}{\eta_2 - \beta_4}\right). \quad (2.39)$$

Since $\beta_3$ and $\beta_4$ solve $f(x) = G(\cdot) - r = 0$, we learn that $f(\beta_3) = f(\beta_4) = 0$, and then

$$(\mathcal{L}V)(x) - rV(x) = \lambda q e^{(x_0-x)\eta_2} \left(K - \frac{\eta_2 e^{x_0}}{1 + \eta_2} - \frac{\eta_2 A e^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 B e^{-\beta_4 x_0}}{\eta_2 - \beta_4}\right). \quad (2.40)$$

Besides by $V(x_0^-) = V(x_0^+)$ and $V'(x_0^-) = V'(x_0^+)$, the following equations can be obtained

$$K - e^{x_0} = A e^{-\beta_3 x_0} + B e^{-\beta_4 x_0}, \quad (2.41)$$

$$-e^{x_0} = -A\beta_3 e^{-\beta_3 x_0} - B\beta_4 e^{-\beta_4 x_0}. \quad (2.42)$$

Hence it can be derived that

$$0 = \left(K - \frac{\eta_2 e^{x_0}}{1 + \eta_2} - \frac{\eta_2 A e^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 B e^{-\beta_4 x_0}}{\eta_2 - \beta_4}\right). \quad (2.43)$$

From above we have verified condition 2.

3. Check if $(\mathcal{L}V)(x) - rV(x) < 0, \forall x < x_0$. For the case of $x < x_0$, we have
derived that

\[
(LV)(x) - rV(x) = -rK - \lambda pe^{-(x_0 - x)\eta_1} \left( K - \frac{\eta_1 e^{x_0}}{\eta_1 - 1} - \frac{\eta_1 A e^{-\beta_1 x_0}}{\eta_1 + \beta_3} - \frac{\eta_1 B e^{-\beta_4 x_0}}{\eta_1 + \beta_4} \right).
\]  

(2.44)

Before calculating \((d)\), we may rewrite \(A\) and \(B\) as

\[
A = e^{\beta_3 x_0} \frac{1 + \beta_4}{\beta_4 - \beta_3} \left( \frac{\beta_4}{1 + \beta_4} K - v_0 \right),
\]

(2.45)

\[
B = e^{\beta_4 x_0} \frac{1 + \beta_3}{\beta_4 - \beta_3} \left( v_0 - \frac{\beta_3}{1 + \beta_3} K \right),
\]

(2.46)

and substitute \(A\) and \(B\) into the \((d)\)-term to obtain

\[
(d) = K - \frac{\eta_1 e^{x_0}}{\eta_1 - 1} - \left[ e^{\beta_3 x_0} \frac{1 + \beta_4}{\beta_4 - \beta_3} \left( \frac{\beta_4}{1 + \beta_4} K - v_0 \right) \right] \frac{\eta_1 e^{-\beta_1 x_0}}{\eta_1 + \beta_3}
- \left[ e^{\beta_4 x_0} \frac{1 + \beta_3}{\beta_4 - \beta_3} \left( v_0 - \frac{\beta_3}{1 + \beta_3} K \right) \right] \frac{\eta_1 e^{-\beta_4 x_0}}{\eta_1 + \beta_4},
\]

(2.47)

\[
= -K \frac{\beta_3 \beta_4}{(\eta_1 + \beta_3)(\eta_1 + \beta_4)} \frac{\eta_1 + \eta_2}{\eta_1}. 
\]

(2.48)

Thus we obtain

\[
(LV)(x) - rV(x) = -rK + \lambda pe^{-(x_0 - x)\eta_1} \cdot K \frac{\beta_3 \beta_4}{(\eta_1 + \beta_3)(\eta_1 + \beta_4)} \frac{\eta_1 + \eta_2}{\eta_1}. 
\]

(2.49)

The requirement \(\eta_1 > 1\) implies that \((LV)(x) - rV(x)\) is an increasing function in \(x\). To show that condition 3 holds, it suffices to demonstrate that \((LV)(x_0-) - rV(x_0-) \leq 0\). Because of the boundedness and continuity of \(V(x)\), the Dominated Convergence Theorem tells us that

\[
(LV)(x_0-) - (LV)(x_0+) = \frac{1}{2} \left( V'(x_0-) - (V'(x_0+)) \right),
\]

\[
= \frac{1}{2} \left( -e^{x_0} - A \beta_3 e^{-\beta_3 x_0} - B \beta_4 e^{-\beta_4 x_0} \right) \leq 0,
\]

which is equivalent to

\[
(LV)(x_0-) - rV(x_0-) - [(LV)(x_0+) - rV(x_0+)] \leq 0,
\]

(2.50)
since we have \( rV(x_0-) = rV(x_0+) \). It is given in condition 2. that \((LV)(x_0+) - rV(x_0+)=0\), then (2.51) becomes

\[
(LV)(x_0-) - rV(x_0-) \leq 0, \tag{2.51}
\]

which tells that condition 3. holds true.

As suggested in Kou and Wang (2001), given that the value function is continuous at \( x_0 \) with \( V(x_0-) = V(x_0+) \), and bounded \( 0 \leq V(x) \leq K \), conditions 4., 5. and 6. can be verified. Therefore we may apply Lemma 2.2.1. to the value function \( V(x) \) and the proof is done.

Similarly perpetual American call option can be treated in the same way. These pricing formulas for perpetual American put seem to be complicated, however they are useful and can be applied to approximate the pricing of finite-horizon American put as in Kou and Wang (2004). Besides Andersen and Andreasen (2000) show that under this model, the curve of implied volatility versus strike price resembles a smile.

Let us have some intuition for Theorem 2.2.2. To price such a path-dependent option, one must study the first passage times while the jump process crosses the boundary. Assume \( l > 0 \) be the boundary. Sometimes the jump process hits the boundary exactly and sometimes it results in an overshoot \( X(\tau_l) - l \) over the boundary. Here \( \tau_l \) denotes the first passage time. To compute \( \tau_l \) analytically, one must know the distribution of the overshoot \( X(\tau_l) - l \). Besides, it is also necessary to learn the dependent structure between the first passage time \( \tau_l \) and the overshoot \( X(\tau_l) - l \). These requirements are possible if the jump size \( Y \) has an exponential-type distribution. These are the reasons why the path-dependent options can have closed-form pricing formulas under the double exponential jump diffusion model.

Moreover we may develop an intuition to the coefficients in Theorem 2.2.2. As mentioned previously, the infinitesimal generator of \( X(t) \) has the form

\[
(LV)(x) = \frac{1}{2}\sigma^2 V''(x) + \left(r - \frac{1}{2}\sigma^2 - \lambda \zeta \right) V'(x) + \lambda \int_{-\infty}^{+\infty} [V(x+y) - V(x)] dF(y). \tag{2.52}
\]

Consider the optimal stopping problem as in (2.10), we know that for all \( x \) in the continuity region, one has

\[
(LV)(x) - rV(x) = 0, \forall x > x_0 = \log(v_0). \tag{2.53}
\]
This is quite similar to the solution of perpetual American put under the geometric Brownian motion model. We may note that only the infinitesimal generator \((\mathcal{L}V)(x)\) is different. In the proof of Theorem 2.2.2., we have seen that for all \(x > x_0\), one has

\[
(\mathcal{L}V)(x) - rV(x) = Ae^{-\beta_3 x} f(\beta_3) + Be^{-\beta_4 x} f(\beta_4),
\]

\[
+ \lambda q e^{(x_0-x)\eta_2} \left( K - \frac{\eta_2 e^{x_0}}{1 + \eta_2} - \frac{\eta_2 Ae^{-\beta_3 x_0}}{\eta_2 - \beta_3} - \frac{\eta_2 Be^{-\beta_4 x_0}}{\eta_2 - \beta_4} \right),
\]

(2.54)

with \(f(x) = G(-x) - r\). To solve (2.53), it suffices to have each of the three terms in the R.H.S. of (2.54) to be zero. Together with the smooth-fit condition, one may have \(A, B, \beta_3\) and \(\beta_4\) and \(v_0\) as shown in Theorem 2.2.2.
Chapter 3

The Small Investor’s Optimal Consumption and Portfolio Choice in an Incomplete Market Driven by a Jump Diffusion Process

In the most important chapter of the thesis, we would show that how can a small investor make optimal consumption and portfolio choice in an incomplete market driven by a jump diffusion process. “Small investor” is in the sense that the actions of the investor cannot affect the market prices.

We would basically follow the work and notation of Bellamy (2001); however, while they consider the investor’s utility based on terminal wealth, we would postulate that investor’s utility is related to both consumption and terminal wealth. We begin with an introduction to the model in section 3.1, and some results would be given in section 3.2.

3.1 The Model

The wealth optimization problems in a complete market have been investigated by many authors, say Merton (1971). Some authors, for instance, Karatzas (1989) considers both consumption and wealth optimization in a complete market with Brownian motion. In the work of Jeanblanc and Pontier (1990), they add jump diffusion to Karatzas’ model in the complete market case. Bellamy (2001) broadens Jeanblanc and
Pontier’s results by constructing the model in an incomplete market driven by discontinuous prices. It is worthy to mention the motivation of Bellamy (2001) here. Bellamy claims that in such an incomplete market driven by discontinuous prices, there exists several equivalent martingale measures, and all of them define a no arbitrage price. He shows that the application of a utility function is a well-adapted method to reduce the range of prices. However, Bellamy considers only the wealth optimization but consumption optimization problem. In this chapter, we would extend the work of Bellamy (2001) by considering the small investor’s utility based on both consumption and terminal wealth.

Set that \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is the probability space, and there are one riskless bond \(B_t\) and one risky asset \(S_t\) in the market. \(B_t\) and \(S_t\) have the following dynamics respectively,

\[
\begin{align*}
    dB_t &= r(t)B_t dt, \\
    dS_t &= S_t [b(t)dt + \sigma(t) dW_t + \varphi(t) dM_t]
\end{align*}
\]

with the riskless interest rate \(r(t)\), rate of return \(b(t)\), volatility of Brownian motion \(\sigma(t)\), and volatility of the compensated martingale of Poisson process \(\varphi(t)\). Besides, \(W\) is the Brownian motion and \(M\) is the compensated martingale of Poisson process \(N\), with intensity \(\lambda\). According to Revuz and Yor (1994), the martingales \(\{W_t; t \geq 0\}\) and \(\{M_t; t \geq 0\}\) are independent under probability measure \(\mathbb{P}\).

Let \((\mathcal{F}_t)_{t \geq 0}\) be the filtration generated by the martingales \(W\) and \(M\) such as

\[
\mathcal{F}_t = \sigma(W_s, M_s; s \leq t).
\]

Then we assume the following conditions hold:

\[
[H] \begin{cases}
    r, b, \sigma, \varphi, \text{ and } \lambda \text{ are bounded deterministic functions.} \\
    \lambda(t) \geq 0, r(t) \geq 0, \forall t \in [0, T]. \\
    \text{volatility of Brownian motion } \sigma(t) \text{ is positive and bounded:} \\
    \exists a_1 \in \mathbb{R}, \exists a_2 \in \mathbb{R}, 0 \leq a_1 \leq \sigma(t) \leq a_2, \forall t \in [0, T]. \\
    \text{volatility of the compensated martingale of Poisson process } \varphi(t) \text{ is bounded:} \\
    \exists c_1 \in \mathbb{R}, \exists c_2 \in \mathbb{R}, -1 \leq c_1 \leq \sigma(t) \leq c_2, \forall t \in [0, T].
\end{cases}
\]
We consider an investor who has both utility $U_1$ based on consumption $c(t)$ and utility $U_2$ based on terminal wealth $X_T$. The utility functions $U_i$, $i = 1, 2$ are defined on $\mathbb{R}^+_*$ and satisfy the properties:

$$\begin{align*}
\text{[H$_u$]} \quad & U_i \text{ is a strictly concave, nondecreasing and } C^2 	ext{ function on } \mathbb{R}^+_* \text{ for } i = 1, 2. \\
& \lim_{l \to +\infty} U'_i(l) = 0; \lim_{l \to 0} U'_i(l) = +\infty.
\end{align*}$$

The assumption [H$_u$] implies that $U'_i(l)$ is continuous, positive, and strictly decreasing on $\mathbb{R}^+_*$ and this allows $U'_i(l)$ to have an inverse $I_i$ on $]0, +\infty[$ such that

$$\begin{align*}
& \lim_{l \to 0} I_i(l) = 0; \lim_{l \to +\infty} I_i(l) = +\infty.
\end{align*}$$

Let $m_t$ be the number of shares of asset $S_t$, then $\pi_t = m_t \times S_t$ is the amount of wealth invested in asset $S_t$. The small investor would allocate his wealth among risky asset $S_t$, riskless bond $B_t$ and consumption $c_t$.

**Definition 3.1.1.** A consumption process $C = \{c_t; 0 \leq t \leq T\}$ is a measurable adapted process with values in $[0, +\infty[$ such that

$$\int_0^T c(t) dt < +\infty \ a.s. \quad (3.4)$$

A portfolio process $\Pi_t = \{\pi_t; 0 \leq t \leq T\}$ is a $\mathcal{F}_t$-predictable process with

$$\int_0^T \pi^2(t) dt < +\infty \ a.s. \quad (3.5)$$

Assume that at time $t = 0$, the small investor has the initial wealth $x$ as $X^x_0 = x$ and there is no money flowing in or out during time interval $[0, T]$, which means that the small investor constructs his asset in a self-financing manner. Let $X_t = \{X^x_t; 0 \leq t \leq T\}$ be the wealth process. Then the change of the wealth $X^x_t$ comes from the change of the asset prices of $B_t, S_t$ and the amount of consumption $c_t$ like

$$dX^x_t = dB_t + m(t) dS_t - c(t) dt. \quad (3.6)$$

Let us substitute $dB_t$ from (3.1) and $dS_t$ from (3.2) into (3.6). After some calculation, the wealth process has the following dynamics
\[ dX_t^{\pi,x} = \left[ \pi_t(b(t) - r(t)) + r(t)X_t^{\pi,x} - c(t) \right] dt + \pi_t \left[ \sigma(t)dW_t + \varphi(t)dM_t \right]. \] 

(3.7)

**Definition 3.1.2.** The consumption-portfolio pair \((c, \pi)\) satisfying (3.4) and (3.5) is admissible for the initial wealth \(x\) if

\[ X_t^{\pi,x} \geq 0, \forall t \in [0,T] \text{ a.s.} \] 

(3.8)

Besides, the set \(\mathcal{A}(t, x)\) is the set of admissible pairs \((c, \pi)\) for the initial wealth \(x\).

Then we come to the value function of the small investor, who distributes his wealth in riskless asset \(B_t\), risky asset \(S_t\) and consumption \(c_t\) and wishes to maximize the expected utility bottomed on the consumption and the terminal wealth. The value function \(V(t, x)\) is defined as

\[ V(t, x) = \sup_{(c, \pi) \in \mathcal{A}(t, x)} E \left[ \int_0^T U_1(c(t)) \, dt + U_2(X_T^\pi) | X_0^\pi = x \right]. \] 

(3.9)

### 3.2 Optimal Consumption and Portfolio Choice

Let us start with an incomplete market consisting of one riskless asset \(B_t\), and one risky asset \(S_t\), namely \((B_t, S_t)\) with

\[ dB_t = r(t)B_t \, dt, \]
\[ dS_t = S_t \left[ b(t) \, dt + \sigma(t) \, dW_t + \varphi(t) \, dM_t \right]. \] 

(3.10)

First we would prove Lemma 3.2.1, which guarantees the boundedness of the value function \(V(t, x)\). This lemma is fundamental for the sequent results.

**Lemma 3.2.1.** The value function of the incomplete market is bounded and satisfies

\[ V(t, x) < +\infty, \forall (t, x) \in [0, T] \times \mathbb{R}^+ \text{ a.s.} \] 

(3.12)

**Proof.** As in Bardhan and Chao (1995) or Bellamy (1999), the optimization problem in the incomplete market can be solved in a completion way, namely by adding the
proper asset process $S_1$ to complete the market as
\[
\begin{align*}
    dS^1(t) &= S^1_1 \left[ b^1(t) dt + \sigma^1(t) dW_t + \varphi^1(t) dM_t \right], \\
    S^1(0) &= s^1(0),
\end{align*}
\]
in which the coefficients $b^1(t), \sigma^1(t)$ and $\varphi^1(t)$ gratify the assumption $[H]$. Once the market is completed by these asset processes $(B_t, S_t, S^1_t), V^c(t, x)$ would be the value function and $\mathcal{A}^c(t, x)$ would be the set of admissible consumption-portfolio pairs for the complete market. It can be derived that every pair $(c, \pi)$ which is admissible in the incomplete market would also be admissible in the complete market; denoted by
\[
(c, \pi) \in \mathcal{A}(t, x) \Rightarrow (c, \pi, 0) \in \mathcal{A}^c(t, x), \text{ which implies } \mathcal{A}(t, x) \subseteq \mathcal{A}^c(t, x).
\]
Then the value functions $V(t, x)$ and $V^c(t, x)$ would have the relationship that
\[
\forall (t, x) \in [0, T] \times \mathbb{R}^+, V(t, x) \leq V^c(t, x).
\]
According to Jeanblanc and Pontier (1990), $V^c(t, x)$ is finite:
\[
\forall (t, x) \in [0, T] \times \mathbb{R}^+, V^c(t, x) < +\infty.
\]
Thus the above two inequalities have proved this lemma. \hfill \square

Before showing the existence of the optimal consumption-portfolio pair $(c^*, \pi^*)$ and characterize it, we assume the value function gratifies the following conditions:
\[
\begin{align*}
    \begin{cases}
        \text{Both } [H] \text{ and } [H_v] \text{ hold.} \\
        \text{$V(t, x)$ is $C^2$ with respect to the second argument $x$.} \\
        \frac{\partial^2 V}{\partial x^2}(t, x) \neq 0, \forall (t, x) \in [0, T] \times \mathbb{R}^+
    \end{cases}
\end{align*}
\]

The assumption $[H_v]$ implies that the function $V(t, x)$ is strictly concave, strictly nondecreasing and therefore $\frac{\partial V}{\partial x}(t, x)$ has an inverse function defined over $\mathbb{R}$. Then we would introduce Lemma 3.2.2 and Lemma 3.2.3, which help to define the equivalent martingale measure in this incomplete market.

**Lemma 3.2.2.** Suppose the assumption $[H_v]$ holds. Then the following equation admits
a unique solution $z_0$ in $\mathbb{R}$ for all $(t, x) \in [0, T] \times \mathbb{R}^+$

$$z + \lambda(t) \varphi(t) - \frac{\lambda(t) \varphi(t)}{V'(t, x)} V' \left[ t, x + \varphi(t) \left( \frac{r(t) - b(t) - z}{\sigma^2(t)} \right) \frac{V'(t, x)}{V''(t, x)} \right] = 0. \quad (3.13)$$

Here, $V'(t, x)$ and $V''(t, x)$ stand for $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V}{\partial x^2}$, respectively.

**Proof.** For all $(t, x) \in [0, T] \times \mathbb{R}^+$, let the function $F(z)$ be defined as

$$F(z) = z + \lambda(t) \varphi(t) - \frac{\lambda(t) \varphi(t)}{V'(t, x)} V' \left[ t, x + \varphi(t) \left( \frac{r(t) - b(t) - z}{\sigma^2(t)} \right) \frac{V'(t, x)}{V''(t, x)} \right].$$

By the assumption $[H_e]$, we know that $V'(t, x) > 0$ and $V''(t, x) < 0$, and therefore

$$\frac{V'(t, x)}{V''(t, x)} < 0; \quad -z \cdot \frac{V'(t, x)}{V''(t, x)} > 0 \text{ for } z > 0.$$

When $z$ increases, $-z \cdot \frac{V'(t, x)}{V''(t, x)}$ also increases and $V'[t, \cdot]$ decreases, which infers that $-V'[t, \cdot]$ strictly nondecreasing in $z$; accordingly, $F(z)$ is a strictly nondecreasing function in $z$. Then we take the Taylor expansion of $V'(t, x + b')$ with $b'$ a small real number as follows

$$V'(t, x + b') = V'(t, x) + V''(t, x)b' + \frac{1}{2}V^{(3)}(t, x)b'^2 + \ldots \quad (3.14)$$

Let us take $b' = \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x)$, and substitute it into (3.14) to obtain

$$V' \left[ t, x + \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x) \right] = V'(t, x) + V''(t, x) \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x) + \frac{1}{2}V^{(3)}(t, x) \left( \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x) \right)^2 + \ldots$$

There exists a real number $\zeta$ such that,

$$V' \left[ t, x + \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x) \right] = V'(t, x) + V''(t, \zeta) \varphi(t) \frac{r(t) - b(t)}{\sigma(t)^2} V'(t, x). \quad (3.15)$$
Now we take \( z = 0 \) in the function \( F(z) \) and have \( F(0) \) as

\[
F(0) = \lambda(t)\varphi(t) - \frac{\lambda(t)\varphi(t)}{V''(t,x)} \left[ t, x + \varphi(t) \left( \frac{r(t) - b(t)}{\sigma^2(t)} \right) \frac{V'(t,x)}{V''(t,x)} \right]. \tag{3.16}
\]

Substitute \( V'[t, \cdot] \) from (3.15) into (3.16), and \( F(0) \) can be written as

\[
F(0) = \lambda(t)\varphi(t) - \frac{\lambda(t)\varphi(t)}{\sigma^2(t)V''(t,x)} \left[ V'(t,x) + V''(t,\zeta)\varphi(t)\frac{r(t) - b(t)}{\sigma^2(t)} \frac{V'(t,x)}{V''(t,x)} \right] = -\frac{\lambda(t)\varphi(t)^2}{\sigma^2(t)V''(t,x)} (r - b)(t)V''(t,\zeta). \tag{3.17}
\]

Since \(-\frac{\lambda(t)\varphi(t)^2}{\sigma^2(t)V''(t,x)} > 0\) and \( V''(t,\zeta) < 0 \), we know that \( F(0) \) and \( (b - r) \) have the same sign. Besides, we take \( z = r - b \), and \( F(r - b) \) is given by

\[
F(r - b) = r - b + \lambda(t)\varphi(t) - \frac{\lambda(t)\varphi(t)}{V''(t,x)} \left[ t, x + \varphi(t) \left( \frac{r - b - (r - b)}{\sigma^2(t)} \right) \frac{V'(t,x)}{V''(t,x)} \right] = r - b + \lambda(t)\varphi(t) - \lambda(t)\varphi(t) = r - b.
\]

From above, we have \( F(r - b) = r - b \). Now we may discuss the two cases for the root of \( F(z) \).

- \( r - b > 0 \):
  - This suggests that \( F(r - b) = r - b > 0 \) and \( F(0) \propto (b - r) < 0 \). Since \( F(z) \) is strictly nondecreasing in \( z \), there must exist some \( z_0 \) in \( \mathbb{R} \) such that
    \[
    0 < z_0 < r - b; F(z_0) = 0.
    \]

- \( r - b < 0 \):
  - In this case we have \( F(r - b) = r - b < 0 \) and \( F(0) \propto (b - r) > 0 \). Since \( F(z) \) is strictly nondecreasing in \( z \), there must exist some \( z_0 \) in \( \mathbb{R} \) such that
    \[
    r - b < z_0 < 0; F(z_0) = 0.
    \]

From the contents above, Lemma 3.2.2 is proved. \( \square \)

In Lemma 3.2.3, we would continue to characterize the equivalent martingale measure in this incomplete market with the help of Lemma 3.2.2.
Lemma 3.2.3. Suppose the assumption $[H_u]$ holds, and let $\chi(t, \cdot)$ denote $V'(t, \cdot)^{-1}$. Then we have these two results.

1. For all $(t, x) \in [0, T] \times \mathbb{R}^+$, there exists some $\gamma^u_t \in ]1, +\infty[ $ such that

$$\chi[t, (1 + \gamma^u_t)^2 V'(t, x)] - x = \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - \lambda(t) \varphi(t) \gamma^u_t) \frac{V'(t, x)}{V''(t, x)}. \quad (3.18)$$

2. The equation (3.18) defines a unique equivalent martingale measure $\mathbb{P}^\gamma$.

Proof. To prove 1. in Lemma 3.2.3, let us define $\gamma^u_t$ in the following way:

$$1 + \gamma^u_t \triangleq \frac{1}{V'(t, x)} V' \left[ t, x + \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - z_0(t, x)) \frac{V'(t, x)}{V''(t, x)} \right], \quad (3.19)$$

where $z_0(t, x)$ is given in Lemma 3.2.2 and satisfies

$$0 = z_0 + \lambda(t) \varphi(t) - \frac{\lambda(t) \varphi(t)}{V'(t, x)} V' \left[ t, x + \varphi(t) \left( \frac{r(t) - b(t) - z_0}{\sigma^2(t)} \right) \frac{V'(t, x)}{V''(t, x)} \right]. \quad (3.20)$$

With the definition of $\gamma^u_t$ in (3.19), we may have $V'(t, \cdot) = (1 + \gamma^u_t) \times V'(t, x)$ and substitute this into (3.20), then (3.20) turns into

$$0 = z_0 + \lambda(t) \varphi(t) - \lambda(t) \varphi(t)(1 + \gamma^u_t), \quad (3.21)$$

which tells us $z_0 = \lambda(t) \varphi(t) \gamma^u_t$. To show that $\gamma^u_t$ is the unique solution of (3.18), we put the definition of $\gamma^u_t$ into the L.H.S. of (3.18):

$$\chi \left[ t, \left( \frac{1}{V'(t, x)} V' \left[ t, x + \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - z_0(t, x)) \frac{V'(t, x)}{V''(t, x)} \right] \right) V'(t, x) \right] - x,$$

$$= \chi \left[ t, V' \left[ t, x + \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - z_0(t, x)) \frac{V'(t, x)}{V''(t, x)} \right] \right] - x,$$

$$= \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - z_0(t, x)) \frac{V'(t, x)}{V''(t, x)},$$

$$= \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - \lambda(t) \varphi(t) \gamma^u_t) \frac{V'(t, x)}{V''(t, x)}, \quad (3.22)$$

with $z_0 = \lambda(t) \varphi(t) \gamma^u_t$. And this shows that $\gamma^u_t$ is the unique solution of (3.18).
For the proof of 2. in Lemma 3.2.3, we may denote $\psi^n$ by
\[
\psi^n = \frac{1}{\sigma}(r(t) - b(t) - \lambda(t)\varphi(t)\gamma^n),
\]
then $\psi^n$ defines a unique equivalent martingale measure such that the related Radon-Nikodym derivative has the form
\[
\frac{dQ}{dP} = \mathcal{E}(\psi^nW)_T\mathcal{E}(\gamma^n M)_T = L^n_T,
\]
where $\mathcal{E}(\psi^nW)_t$ and $\mathcal{E}(\gamma^n M)_t$ are the Doléans-Dade’s exponentials defined as
\[
\mathcal{E}(\psi^nW)_t = \exp\left[\int_0^t \psi_s dW_s - \frac{1}{2} \int_0^t \psi_s^2 ds\right],
\]
\[
\mathcal{E}(\gamma^n M)_t = \exp\left[\int_0^t \ln(1 + \gamma_s) dM_s - \int_0^t \lambda(s) (\gamma_s - \ln(1 + \gamma_s)) ds\right].
\]

Thanks to Lemmas 3.2.2 and 3.2.3, now we may come to the main result of this chapter.

**Theorem 3.2.4.** Suppose that the assumption $[H_v]$ holds and the initial wealth $x$ is strictly positive. Besides, utility function $U_i$ satisfies
\[
\forall l, \exists C > 0, \exists p \in \mathbb{N} \text{ such that } |U_i(l)| \leq C(1 + |l|^p), \text{ for } i = 1, 2.
\]

Then the following two results are given.

1. There exists an optimal portfolio $\pi^*$ of the form
\[
\pi^*(t, x) = \frac{r(t) - b(t) - z_0(t, x)}{\sigma^2(t)} \times \frac{V'(t, x)}{V''(t, x)},
\]

2. The optimal consumption $c^*_t$ and optimal wealth $X^*_t$ are given by
\[
c^*_t \triangleq I_1(g^*L^*_T),
\]
\[
X^*_t \triangleq I_2(g^*L^*_T),
\]
where \( y^* \) is the optimal Lagrangian coefficient defined by

\[
E \left[ \int_0^T R(t) L_t^\gamma u I_1 (y^* L_t^\gamma) dt + R(T) L_T^\gamma u I_2 (y^* L_T^\gamma) \right] = x. \tag{3.31}
\]

Additionally, \( R(t) = \exp(-\int_0^t r(s) ds) \) denotes the discount coefficient; \( \gamma^u \) and \( L_t^\gamma u \) come from Lemma 3.2.3.

**Proof.** As in Bellamy (2001), here we adopt a verification theorem, which can be found in Karatzas and Shreve (1988) for continuous assets, or in Jeanblanc and Pontier (1990) for discontinuous assets. The main idea of the proof is that, assume there is a function \( G \) which is in \( C^{1,2} \) and satisfies the polynomial growth condition

\[
\forall t \in [0, rT], \forall l, |G(trl)| \leq C(1 + |l|^p); C > 0 \text{ and } p \in \mathbb{N}. \tag{3.32}
\]

We may find an admissible consumption-portfolio pair \((c_t^*, \pi_t^*)\) such that \( G(tr c_t^* r \pi_t^*) \) forms the solution of Hamilton-Jacobi-Bellman (HJB) equation with boundary condition. If \( G \) is regular enough, \( G \) would then be the value function and \((c_t^*, \pi_t^*) \) would be the optimal consumption-portfolio pair.

In the beginning, we find out the HJB equation for our model with incomplete market, and then we manifest that this HJB equation has a unique solution, namely \((c_t^*, \pi_t^*)\) in Theorem 3.2.4. Let \( \delta \) be in \([0, T]\). For \( t = \delta \), the investor’s wealth equals to \( x \). The wealth process corresponding to the portfolio \( \pi \) is denoted by \( X_{\delta,x,\pi}^{t} ; \delta \leq t \leq T \) with dynamics below

\[
dX_{\delta,x,\pi} = \left[ \pi_t (b(t) - r(t)) + r(t) X_{\delta,x,\pi}^t - c(t) \right] dt + \pi_t [\sigma(t) dW_t + \varphi(t) dM_t],
\]

\[
X_{\delta,x,\pi}^\delta = x.
\]

By the dynamic programming principle from Fleming and Rishel (1975) or Proposition 5.1 in Jeanblanc and Pontier (1990), we have

\[
\forall (\delta, x) \in [0, T] \times \mathbb{R}^+, \forall t \in [\delta, T], V(\delta, x) = \sup_{(c,\pi) \in A(t,x)} E[V(t, X_{t}^{\delta,x,\pi})]. \tag{3.33}
\]

So the HJB equation can be written as

\[
\frac{\partial V(t, x)}{\partial t} + \sup_{(c,\pi) \in A(t,x)} \mathcal{L}^X V(t, x) = 0, \tag{3.34}
\]
The definition of (3.38) can be rewritten in the subsequent way:

\[\mathcal{L}^X V(t, x) = U_1(c) + [xr(t) - c(t) - \pi (b(t) - \lambda(t)\varphi(t) - r(t))] \frac{\partial V}{\partial x}(t, x) + \frac{1}{2}(\sigma\pi)^2 \frac{\partial^2 V}{\partial x^2}(t, x) + \lambda(t) \left[V(t, x + \pi\varphi(t)) - V(t, x)\right].\]  

(3.35)

At time \(T\), we obtain the boundary condition

\[\forall x \in \mathbb{R}^+, G(T, x) = U_2(x).\]  

(3.36)

Making the partial derivatives of (3.35) with respect to \((c, \pi)\) and we may acquire

\[
\begin{align*}
\frac{\partial \mathcal{L}^X V(t, x)}{\partial c} &= 0 \quad \Rightarrow \quad U'_1(c) - \frac{\partial V(t, x)}{\partial x} = 0, \\
\frac{\partial \mathcal{L}^X V(t, x)}{\partial \pi} &= 0 \quad \Rightarrow \quad (b(t) - \lambda(t)\varphi(t) - r(t)) \frac{\partial V}{\partial x}(t, x) + \sigma^2 \pi \frac{\partial^2 V}{\partial x^2}(t, x) \\
&\quad \quad + \lambda(t)\varphi(t) \frac{\partial V}{\partial x}(t, x + \pi\varphi(t)) = 0.
\end{align*}
\]

(3.37)

(3.38)

By Lemma 3.2.2 together with Lemma 3.2.3, we learn that \(z_0 = \lambda(t)\varphi(t)\gamma^n_t\), and put \(z_0 - z_0 = \lambda(t)\varphi(t)\gamma^n_t - \lambda(t)\varphi(t)\gamma^n_t\) into the bracket of the first term in (3.38), then (3.38) can be rewritten in the subsequent way:

\[
\begin{align*}
&\frac{\partial \mathcal{L}^X V(t, x)}{\partial c} = 0 \quad \Rightarrow \quad U'_1(c) - \frac{\partial V(t, x)}{\partial x} = 0, \\
&\frac{\partial \mathcal{L}^X V(t, x)}{\partial \pi} = 0 \quad \Rightarrow \quad (b(t) - \lambda(t)\varphi(t) - r(t)) \frac{\partial V}{\partial x}(t, x) + \sigma^2 \pi \frac{\partial^2 V}{\partial x^2}(t, x) \\
&\quad \quad + \lambda(t)\varphi(t) \frac{\partial V}{\partial x}(t, x + \pi\varphi(t)) = 0, \\
&\quad \quad + \sigma^2 \pi \frac{\partial^2 V}{\partial x^2}(t, x) + \lambda(t)\varphi(t) \frac{\partial V}{\partial x}(t, x + \pi\varphi(t)) = 0, \\
&\quad \quad + \lambda(t)\varphi(t) \left[-(1 + \gamma^n_t)V''(t, x) + V'(t, x + \pi\varphi(t))\right] = 0.
\end{align*}
\]

(3.39)

(3.40)

(3.41)

The definition of \(\gamma^n_t\) tells that \(1 + \gamma^n_t = \frac{1}{V'(t, x)} V' \left[t, x + \frac{\varphi(t)}{\sigma^2(t)} (r(t) - b(t) - z_0(t, x)) \frac{V'(t, x)}{V'(t, x)}\right],\)

31
and put this into (3.41), then (3.42) is obtained

\[
(b(t) - r(t) + z_0) V'(t, x) + \sigma^2 \pi V''(t, x) \\
- \lambda(t) \varphi(t) \left\{ -V \left[ t, x + \varphi(t) \left( \frac{r(t) - b(t) - z_0(t, x)}{\sigma^2(t)} \right) \frac{V'(t, x)}{V''(t, x)} \right] + V'(t, x + \pi \varphi(t)) \right\} = 0.
\]

(3.42)

Since \(V'\) is a strictly nondecreasing function, for the last term of (3.42) to be zero, we must have

\[
\pi(t) = \frac{r(t) - b(t) - z_0(t, x)}{\sigma^2(t)} \frac{V'(t, x)}{V''(t, x)} = \pi^*.
\]

Intuitively we might guess this is the solution of (3.42) and we check it by substituting \(\pi^*\) back into (3.42). Indeed, we can claim that \(\pi^*\) is the solution, and it yields a supremum for \(\mathcal{L}^X V(t, x)\). Of course \((c^*, \pi^*) \in \mathcal{A}(t, x)\) is prerequisite. This proves 1. in Theorem 3.2.4.

Then we continue to prove 2. in Theorem 3.2.4. Let \(\mathbb{P}^{\gamma^*}\) be the equivalent martingale measure given by Lemma 3.2.3. To complete the market as in the proof of Lemma 3.2.1, we may find the appropriate asset \(S^1(t)\) such that,

\[
dS^1(t) = S^1_t \left[ b^1(t) dt + \sigma^1(t) dW_t + \varphi^1(t) dM_t \right],
\]

and the market defined by \((B_t, S_t, S^1_t)\) would be complete, with the unique equivalent martingale measure \(\mathbb{P}^{\gamma^*}\). The work of Jeanblanc and Pontier (1990) declares that the optimization problem in the complete market \((B_t, S_t, S^1_t)\) has a unique solution. Let \((\tilde{\pi}^*, \tilde{\pi}^*_1), \tilde{c}^*, \tilde{X}^*_T, \tilde{V}\) be the optimal portfolios, the optimal consumption, the optimal wealth and the value function in the complete market respectively. Refer to Bellamy (1999) or Jeanblanc and Pontier (1990), we may have some established results.

1. By Proposition 3.2 in Jeanblanc and Pontier (1990), the optimal consumption \(\tilde{c}^*_t\) satisfies \(\tilde{c}^*_t = I_1(y^* L^\gamma_{1t})\) and similarly, the optimal wealth \(\tilde{X}^*_T\) satisfies \(\tilde{X}^*_T = I_2(y^* L^\gamma_{2T})\), where \(y^*\) is the Lagrangian coefficient defined in the budget constraint.

\[
E \left[ \int_0^T R(t) L^\gamma_{1t} I_1(y^* L^\gamma_{1t}) dt + R(T) L^\gamma_{2T} I_2(y^* L^\gamma_{2T}) \right] = x.
\]

(3.44)

2. Since the initial wealth is supposed to be strictly positive, we have \(\tilde{X}^*_T \geq 0,\)
\( \forall (t, x) \in [0, T] \times \mathbb{R}^+ \).

3. \( \tilde{V}(t, x) \) satisfies the polynomial growth condition:

\[
\forall x, \exists C > 0, \exists p \in \mathbb{N} \text{ such that } |V(t, x)| \leq C(1 + |x|^p), \forall t \in [0, T]. \tag{3.45}
\]

It is not surprising since both \( U_1 \) and \( U_2 \) fulfill the polynomial growth condition.

4. The optimal portfolio \((\tilde{\pi}^*, \tilde{\pi}_1^*)\) is admissible. As suggested in Bellamy (2001), Proposition 87 and Lemma 7 in Bellamy (1999) claim that the optimal portfolio in incomplete market equals to its counterpart in complete market, denoted by \( \tilde{\pi}^* = \pi^* \), since the share of the newly added portfolio is zero, which means \( \tilde{\pi}_1^* = 0 \). They also deduce that the optimal wealths and the value functions in both complete and incomplete market are the same:

\[
\tilde{X}_t^x = X_t^{\pi^*, x}, \tilde{V}(t, x) = V(t, x), \forall (t, x) \in [0, T] \times \mathbb{R}^+. \tag{3.28}
\]

Finally, from 2. and 4., we deduce that \( \tilde{X}_t^x = X_t^{\pi^*, x} \geq 0, \forall (t, x) \in [0, T] \times \mathbb{R}^+ \). In addition, \( \pi^* \) described (3.28) is admissible, and \( V(t, x) \) satisfies the polynomial growth condition because of 3. \( \square \)

It is worthy to mention that when there is no jump term, the optimal portfolio in (3.28) turns into

\[
\pi^*(t, x) = \frac{r(t) - b(t)}{\sigma^2(t)} \times \frac{V'(t, x)}{V''(t, x)}. \tag{3.46}
\]

We may note that the term \( z_0(t, x) = \lambda(t) \psi(t) \gamma^u(t) \) in (3.28) disappears here. This is reasonable since \( z_0(t, x) \) is contributed by the jump diffusion process. Besides, with the jump term, we have the risk premium as

\[
b(t) + z_0(t, x) - r(t) \quad \sigma(t). \tag{3.47}
\]

When the jump part is removed, the risk premium becomes

\[
b(t) - r(t) \quad \sigma(t). \tag{3.48}
\]

We may apply Theorem 3.2.4. to a simple example as below. Suppose the small investor has constant relative risk aversion (CRRA) utility functions to the consumption
\( c_t \) and the terminal wealth \( X_T \), and the utility functions can be written as

\[
U_1(c_t) = \frac{c_t^{1-a} - 1}{1-a} \quad \text{for } a > 0 \text{ and } a \neq 1; \quad (3.49)
\]

\[
U_2(X_T) = \frac{X_T^{1-b} - 1}{1-b} \quad \text{for } b > 0 \text{ and } b \neq 1. \quad (3.50)
\]

After making the first order differentiation, we have

\[
U'_1(c_t) = (c_t)^{-a} = y^* L_t^{-a} \quad \text{and} \quad U'_2(X_T) = (X_T)^{-b} = y^* L_T^{-b}. \quad (3.51)
\]

This implies that

\[
c_t^* = \left(y^* L_t^{-a}\right)^{-\frac{1}{a}} \quad \text{and} \quad X_T^* = \left(y^* L_T^{-b}\right)^{-\frac{1}{b}}. \quad (3.52)
\]

To solve the Lagrangian coefficient \( y^* \), we may put \( c_t^* \) and \( X_T^* \) back into the budget constraint as following

\[
E \left[ \int_0^T R(t) L_t^{-a} \left(y^* L_t^{-a}\right)^{-\frac{1}{a}} dt + R(T) L_T^{-a} \left(y^* L_T^{-a}\right)^{-\frac{1}{a}} \right] = x. \quad (3.53)
\]
Chapter 4

Summary

This chapter is the summary of the thesis. In Chapter 1, we have seen the famous Black-Scholes model. However, as mentioned in Kou and Wang (2001), this model with constant mean return rate and constant volatility, can not give a satisfactory explanation to some empirical puzzles, say volatility smile and asymmetric leptokurtic return distribution. Thus many researches are conducted to modify the Black-Scholes model, and the double exponential jump diffusion model is one the proposed models, which we discuss in Chapter 2. Besides, we show the pricing formula of perpetual American put option under the double exponential jump diffusion model. Moreover, one can obtain the closed form solutions of Laplace transforms for lookback and barrier options under this model, as demonstrated in Kou and Wang (2001). The reason why this jump diffusion model can lead to the analytical solutions for path-dependent options, is that the exponential distribution has specific features. Namely, the products of two exponential distributions is still an exponential distribution whose integration is solvable in many cases. For some empirical studies, Metayer (2003) discusses about the modelling of risky bond prices under such model. Besides, Cont and Tankov (2004) provides some calibration for the option pricing formula in jump diffusion model. In Chapter 3, we show that how can a small investor make the optimal portfolio and consumption choice in an incomplete market which is driven by a jump diffusion process. Since the market is not complete, there exist several equivalent martingale measures compatible with non-arbitrage price. As shown in the context, the utility function can be used to define the equivalent martingale measure in a proper way. We give a simple example of CRRA utility function at the end of Chapter 3. One can continue to apply the theory to other examples, say log utility or exponential utility functions.
Bibliography


Affidavit:

I hereby confirm that I prepare this thesis independently, by exclusive reliance on the literature and tools indicated therein. This thesis has not been submitted to any other examination authority in its current or an altered form, and it has not been published.

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